Equations, Constants and Examples

Stefan-Boltzmann Equation: Relates Temperature and Power Flux

$F = \varepsilon \cdot \sigma \cdot T^4$

F = Radiant Flux (irradiance) in W/m^2
ε = emissivity constant (unitless) (≈1 for perfect blackbody radiator, such as a star)
σ = Stefan-Boltzmann constant = 5.67×10^{-8} $W/(m^2K^4)$
T = Temperature in Kelvin (note: K = °C + 273.15°)

Example: If the area of the tungsten filament in a 100 W incandescent light bulb is 0.26 cm^2, and the emissivity is 0.36, what is the temperature of the filament?

Solution:
First, let's solve the equation for T, since that is what we are finding:
$F = \varepsilon \cdot \sigma \cdot T^4$
$F/(\varepsilon \cdot \sigma) = T^4$
$(F/(\varepsilon \cdot \sigma))^{1/4} = T$
Next, let's get F. We know the power is 100 W, but we need it in terms of W/m^2. The area is 0.26 cm^2, which is 2.6×10^{-5} m^2. So F is equal to 100 W/(2.6×10^{-5} m^2) = 3.846×10^6 W/m^2
Now we plug everything in:
$(F/(\varepsilon \cdot \sigma))^{1/4} = T$
$(3.846 \times 10^6\ W/m^2)/(0.36 * 5.67 \times 10^{-8}\ W/(m^2K^4))^{1/4}$
(check the units!) W and m^2 both cancel, leaving $(K^4)^{1/4} = K$ Good!
T = 3705 K = 3432° C

Solar Constant: Radiant Flux at 1 Astronomical Unit, perpendicular to the sun's rays

$S_c = L_{sun}/(4\pi \cdot d^2)$

L_{sun} = Luminosity of the Sun ≈ 3.84×10^{26} W
note:
$L_{sun} = \varepsilon \cdot \sigma \cdot T_{sun}^4 \cdot \text{Surface Area}_{sun}$,
where $\varepsilon = 1$, T_{sun} = 5778 K, and Surface Area_{sun} = 6.0877×10^{18} m^2
(try it!)
d = Earth-Sun distance in meters ≈ 1.49×10^{11} m
S_c = Solar constant ≈ 1370 W/m^2 (try it!- you won't get this exactly, but it should be close. It won't be exact because the sun is not a 100% perfect black body radiator, luminosity varies, d varies, there is dust in space that absorbs some radiation, etc., etc.)

Example: What is the solar constant on Venus, with a distance of 1.08×10^{11} m?
Solution:
This is straight forward, plug and chug…
$S_{c,Venus} = 3.84 \times 10^{26}\ W/(4\pi \cdot (1.08 \times 10^{11}\ m)^2)$
(check the units!) W in numerator, m^2 in denominator, so W/m^2 Good!
$S_{c,Venus}$ ≈ 2620 W/m^2 (Venus is closer to the sun, so it receives more energy than the Earth)

Solar Insolation: average radiant flux received by all of Earth (Top of Atmosphere)

$S_I = S_c \cdot \pi \cdot r_{Earth}^2$/Surface Area$_{earth}$

note: S_c times the area of Earth's shadow divided by the total surface area of the Earth

$S_I = S_c \cdot \pi \cdot r_{Earth}^2 / (4 \cdot \pi \cdot r_{Earth}^2)$

$S_I = S_c/4$

S_I = Solar insolation ≈ 343 W/m^2

S_c = Solar constant ≈ 1370 W/m^2

r_{earth} = radius of the Earth = 6.371×10^6 m

Example: What is the solar insolation on Venus, with a distance of 1.08×10^{11} m?

Solution:

From above, $S_{c,Venus}$ ≈ 2620 W/m^2

So, $S_{I,Venus}$ = (2620 W/m^2)/4 = 655 W/m^2

Lambert's Cosine Law: Power Flux of Sun at an angle versus horizontal (Projection Effect)

$S_A = S_c \cdot \cos(Z) = S_c \cdot \sin(a)$

S_A = Amount of Solar radiation on a surface angled to the Sun (W/m^2)

S_c = Solar Constant ≈ 1370 W/m^2

Z = Zenith angle (angle of the Sun and to straight up overhead)

a = altitude angle (angle from the Sun to a flat horizon) note: Z + a = 90°

Example: What amount of Solar radiation when the Sun is at 30° above the horizon?

Solution:

S_A = 1370 W/m^2 · sin(30°) = 1370 W/m^2 · 0.5 = 685 W/m^2

note:

Excel trigonometry functions are defaulted to work in Radians, not degrees. The conversion factor is:

Radians = Degrees°·π /180°

Degrees° = 180°·Radians/π

Therefore, to enter a function into Excel when using degrees, use:

=SIN(*degrees**PI()/180)

Example: If the Solar radiation is 200 W/m^2, what is the zenith angle?

Solution:

First solve the equation for Z, the zenith angle.

$S_A = S_c \cdot \cos(Z)$

$S_A/S_c = \cos(Z)$

$\cos^{-1}(S_A/S_c) = Z$

Now plug in the numbers:

$\cos^{-1}$((200 W/m^2)/(1370 W/m^2)) = Z   In Excel, the function is =ACOS(…)

Z = 81.61°

This makes sense because 200 W/m^2 is very little energy, so the Sun must be near the horizon; Z is big.

note: Again, Excel uses Radians, so remember to convert the answer back to degrees.

Zenith Angle: Given a time, date, and location, determines the Sun Zenith Angle

$\cos(Z) = \sin(a) = \sin(\phi)\cdot\sin(\delta) + \cos(\phi)\cdot\cos(\delta)\cdot\cos(h)$

Z = zenith angle (angle of the Sun and to straight up overhead)
a = altitude angle (angle from the Sun to a flat horizon) note: Z + a = 90°
ϕ = latitude
δ = solar declination (the latitude where the sun is directly overhead at noon)
h = hour angle= ±15° for every hour -/+ noon

time	h
1000	30°
1100	15°
1200	0
1300	-15°
1400	-30°

Example:
What is the Zenith angle at noon on September 22 at 40°N?
Solution:
We know h = 0, because the time is noon
We also know that δ = 0 because September 22 is the Autumnal Equinox when the Sun is perpendicular to the equator (latitude = 0°).

sin(0) = 0 so the first addend of the equation is equal to 0.
cos(0) = 1 so the second addend simplifies to $\cos(\phi)$

So we get:
$\cos(Z) = \cos(\phi) = \cos(40°)$
so Z = 40°

Example:
What is the Zenith angle at noon on December 22 at 40°N?
Solution:
We know h = 0, because the time is noon
We also know that δ = 23.5 because December 22 is the Winter Solstice when the Sun is perpendicular to the equator (latitude = 23.5° S). Because this is the southern hemisphere and we are in the north in this problem, we think of it as -23.5°, because it is in the other hemisphere.

$\cos(Z) = \sin(\phi)\cdot\sin(\delta) + \cos(\phi)\cdot\cos(\delta)\cdot\cos(h)$
$\cos(Z) = \sin(40)\cdot\sin(-23.5°\cdot\pi/180) + \cos(40°\cdot\pi/180)\cdot\cos(-23.5°\cdot\pi/180)\cdot 1$

You can calculate this here, and it will give you the right answer, but check this out:
$\sin(A)\cdot\sin(B) + \cos(A)\cdot\cos(B) = \cos(A - B)$ It is a cool trig identity!

$\cos(Z) = \cos(40° - (-23.5°)) = \cos(63.5°)$
So Z = 63.5°
Sanity check: when Z is 63.5°, the altitude of the Sun is 26.5° above the horizon. This makes sense for the Winter Solstice in the Northern Hemisphere.
And we found that at noon at all latitudes, cos(h)= 1 and therefore: $Z = \phi - \delta$

Day Length: The length of the day for a given latitude and date

$DL = (24/\pi)\cdot\cos^{-1}(-\tan(\phi)\cdot\tan(\delta))$
$DL = (24/180)\cdot\cos^{-1}(-\tan(\phi)\cdot\tan(\delta))$ for when $\cos^{-1}(x)$ is in degrees

DL = day length (hours)
ϕ = latitude
δ = solar declination (the latitude where the sun is directly overhead at noon)

Example:
What are the maximum and minimum day lengths for College Park MD at 39° N?

Solution:
We know that the maximum day length is on the summer solstice and the solar declination is 23.5°, the Tropic of Cancer. The minimum day length is on the winter solstice and the solar declination is -23.5°, the Tropic of Capricorn. We use a negative because it is in the opposite hemisphere. From here, we (carefully) plug and chug.

$DL_{max} = (24/\pi)\cdot\cos^{-1}(-\tan(39°\cdot\pi/180)\cdot\tan(23.5°\cdot\pi/180))$
$DL_{max} = (24/\pi)\cdot\cos^{-1}(-.3521)$
$DL_{max} = (24/\pi)\cdot 1.93$
DL_{max} = 14.75 hours = 14 hours and 45 minutes

$DL_{min} = (24/\pi)\cdot\cos^{-1}(-\tan(39)\cdot\tan(-23.5))$
DL_{min} = 9.25 hours = 9 hours and 15 minutes

Example:
It is the Winter Solstice and the day is 20 hours long. What is my latitude?

Solution: First, we need to solve the equation for latitude.
$DL = (24/\pi)\cdot\cos^{-1}(-\tan(\phi)\cdot\tan(\delta))$
$(DL\cdot\pi/24) = \cos^{-1}(-\tan(\phi)\cdot\tan(\delta))$
$\cos(DL\cdot\pi/24) = -\tan(\phi)\cdot\tan(\delta)$
$-\cos(DL\cdot\pi/24)/\tan(\delta) = \tan(\phi)$
$\tan^{-1}[-\cos(DL\cdot\pi/24)/\tan(\delta)] = \phi$

Because it is the Winter Solstice, we know that the solar declination angle is at the Tropic of Capricorn, 23.5° S. We also know that our latitude is on the Southern Hemisphere because the day is longer than 12 hours and it is the Winter Solstice. In the Northern Hemisphere, the day lengths are all shorter than 12 hours in winter. Because the latitude is in the same hemisphere as the declination, we can use a positive sign for the declination angle, as long as we remember we are in the Southern Hemisphere.

$\tan^{-1}[-\cos(20\cdot\pi/24)/\tan(23.5°\cdot\pi/180)] = \phi$
$\tan^{-1}[-\cos(2.62)/\tan(0.41)] = \phi$
$\tan^{-1}[2.17] = \phi$
$1.11*180°/\pi = \phi$ note: Excel ATAN() function returns Radians, so I convert to degrees
$\phi = 63.3°$ S

Sensitive Heat: Thermal energy related to temperature increase of a material

$Q_{SH} = m \cdot c \cdot (\Delta T)$

Q_{SH} = Thermal energy (J)
m = mass of material (kg)
c = specific heat capacity (material dependent) J/(kg·K)
ΔT = Change in Temperature (K)

Latent Heat: Energy related to phase transition (absorbed or released)

$Q_{LH} = m \cdot L$

Q_{LH} = Latent heat energy (J)
m = mass of material (kg)
L = latent heat capacity (material dependent, phase shift dependent) J/kg

Example:
How much energy is required to raise the temperature of 10 kg of granite from 100° C to 200° C, if the specific heat capacity of granite is 190 J/(kg·K)?

Solution:
No phase change takes place, so this is just a change in sensitive heat. The change in temperature is 200-100 = 100 K. Remember that a K degree is the same as a °C.

$Q_{SH} = m \cdot c \cdot (\Delta T)$
$Q_{SH} = 10\ \text{kg} \cdot 190\ \text{J/(kg·K)} \cdot 100\ \text{K}$ Do the units work out? Yes, the kg and the K both cancel.
$Q_{SH} = 190{,}000\ \text{J}$

Example:
How much power is required to completely evaporate 2 liters of water stored at room temperature 18° C in 10 minutes if the specific heat capacity of water is 4187 J/(kg·K) and the latent heat of evaporation is 2,465,560 J/kg?

Solution:
First, we need the mass of the water. Conveniently 1 liter of water is equal to 1 kg of mass, so we have 2 kg of water. Next, the temperature difference: water boils at 100° C, so the difference is 82° C = 82 K.

Sensible Heat Part:	Latent Heat Part:
$Q_{SH} = m \cdot c \cdot (\Delta T)$	$Q_{LH} = m \cdot L$
$Q_{SH} = 2\ \text{kg} \cdot 4187\ \text{J/(kg·K)} \cdot 82\ \text{K}$	$Q_{LH} = 2\ \text{kg} \cdot 2465560\ \text{J/kg}$
$Q_{SH} = 686{,}668\ \text{J}$	$Q_{LH} = 4{,}931{,}120\ \text{J}$

Add them: $Q = Q_{SH} + Q_{LH} = 686668\ \text{J} + 4931120\ \text{J} = 5{,}617{,}788\ \text{J}$
But we're not done yet! We need this in terms of power (W = J/s). We have 10 minutes to boil all the water, so that is 600 seconds.
$P = 5617788\ \text{J}/600\ \text{s} = 9363\ \text{W}$

Pressure: Ideal Gas Law

$P = R \cdot T \cdot m/V$

P = Pressure = J/m^3= $N \cdot m/m^3$ = N/m^2 = Pa (Pascal) = .01 millibar (mb)
R = Dry gas constant 287 $J/(kg \cdot K)$
T = Temperature (K)
m = mass (kg)
V = Volume in (m^3)

Example:
Sea level has an average pressure of 101325 Pa. What is the density of the atmosphere if Earth's average temperature is 15° C?

Solution:
First, solve the equation for density, which is m/V:
$P/(R \cdot T) = m/V$

Now, convert the temperature to K: 15+273.15 = 288.15 K, and solve.
101325 Pa/(287 J·288.15 K) = 1.22 kg/m^3

Example:
Assume the pressure and mass remains constant and the temperature decreases by 10° C. What happens to the volume?

Solution:
Let's solve for V:

$P = R \cdot T \cdot m/V$
$V= T \cdot (R \cdot m/P)$
$V= T \cdot \text{constant}$
$V \propto T$
If temperature goes up, then so does the volume (if pressure and mass are constant).

Pressure Gradient Force

$PGF = [(P_H/V_H)- (P_L/V_L)]/d$
$PGF = m\cdot[(P_H/\rho_H)-(P_L/\rho_L)] /d$

PGF = Pressure Gradient Force (N), Force created by difference in pressure between two locations
P = Pressure (Pa) (both High and Low)
V = Volume (m^3) (both High and Low)
ρ = Difference in air density (kg/m^3)
d = distance between points (m)

Coriolis Effect: "Fictitious Force" Apparent force from rotation of Earth

$CE = 2\cdot m\cdot\Omega\cdot\sin(\phi)\cdot v$ (This is different from the text- I inserted mass to make it a force)

CE = Coriolis Effect (N)
m = mass of air (kg)
Ω = Earth angular velocity = $2\pi/86400$ sec = $7.27 \times 10^{-5}\ s^{-1}$
ϕ = Latitude
v = wind velocity (m/s)

Centrifugal Effect: "Fictitious Force" Apparent force from a moving reference frame

$CF = m\cdot v^2/r$ (This is different from the text- I inserted mass to make it a force)

CF = Centrifugal Force (N)
m = mass of air (kg)
v = wind velocity (m/s)
r = radius of curvature of motion (m)

Total Force on Air to Produce Wind

Surface Wind = PGF + CE + CF + Fr

Surface Wind = Force of air movement (N)
PGF = Pressure Gradient Force (N)
CE = Coriolis Effect (N), Perpendicular to trajectory; to the right for N.H., and to the left for S. H.
CF = Centrifugal Force (N), opposite direction as CE.
Fr = Friction (N), opposite direction as trajectory. Empirically measured.

Steady State/ Residence Time

Given $f_{in} = f_{out,}$
$\tau = N/f_{in}$

τ = Residence time
N = Size of reservoir (mass, volume, or number)
f_{in} = the flow into the reservoir (mass, volume or number)/unit time
f_{out} = the flow out of the reservoir (mass, volume or number)/unit time

Example:
To keep a lake stocked with fish, 1000 new fish from the hatchery are added each year. The fish population is stable at 15,000. What is the life expectancy of the fish in the lake?

Solution:
$\tau = N/f_{in}$
τ = 15,000 fish/ (1000 fish/year) = 15 years

Example:
Undergrad enrolment at the University of Maryland is 22,933. Every year, UMD admits 4100 new Freshmen and 2100 transfer students. Assume that these numbers are relatively constant each year. What is the average time an undergraduate spends at UMD?

Solution:
$\tau = N/f_{in}$
τ = 22933 students/ (4100+2100 students/year) = 3.7 years
(Note, this isn't 4 years because some transfer students come in with credits. Also some students drop out early. Other students take longer than 4 years. So on average, it is 3.7 years).

Example:
A rancher adds 500 calves to his herd each year, and sells off 500 mature head of cattle each year. He sells cattle when they are 4 years old. How big is his heard.

Solution:
$f_{in} = f_{out}$ = 500 head/year
$N = f_{in} \cdot \tau$
N = 500 head/year × 4 years = 2000 head of cattle

Conversion of mass CO_2 to mass C

m CO_2 = (44/12)* m C

m CO_2 = mass carbon dioxide
m C = mass carbon

Conversion is based on molecular mass.